JOURNAL OF APPROXIMATION THEORY **56**, 13–29 (1989)

The Relative Distance of J. D. Pryce in $\mathbb{R}^3_\infty$

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Communicated by Oved Shisha

Received July 12, 1985; revised May 24, 1988

Continuing the work of F. W. J. Olver, J. D. Pryce defined a new measure of relative error for vectors. His function $\rho(x, y)$, where x, y are nonzero vectors in a Banach space E, turns $E\backslash\{0\}$ into a metric space; $\rho(x, y)$ is asymptotically equivalent to $\rho_0(x, y) = \|x-y\|/\|x\|$. Using previous results on $\mathbb{R}^2_\infty$ we reduce the computation of $\rho(x, y)$ in $\mathbb{R}^3_\infty$ to pure algebraic manipulations. Quite surprisingly, the distance between two opposite points in $\mathbb{R}^2_\infty$ may be strictly larger than the distance calculated in $\mathbb{R}^3_\infty$ after embedding $\mathbb{R}^2_\infty$ in $\mathbb{R}^3_\infty$ in the obvious way.

Introduction and Notation

Let E be any Banach space. A path C in E is a mapping $t \mapsto x(t)$ defined on a compact interval $a \leqq t \leqq b$ in $\mathbb{R}$ and piecewise smooth in the norm topology; that is, there are points $a = t_0 < t_1 \cdots < t_N = b$ such that, for $1 \leqq i \leqq N$, $x(t)$ has a two-sided derivative in (t_{i-1}, t_i), a right derivative at t_{i-1}, a left derivative at t_i, and the resulting two- or one-sided derivative thus defined in $[t_{i-1}, t_i]$ is continuous there. Any reparametrization $s \mapsto x(\phi(s))$ with ϕ strictly increasing and ϕ and its inverse function piecewise smooth, is regarded as defining the same path C. If $x(a) = A_0$, $x(b) = A_1$, then C is said to connect A_0 with A_1. When $E = \mathbb{R}^n$ we write $C(t)$ instead of $x(t)$ to avoid confusion with the coordinate functions.

If C is a path from A_0 to A_1 we define the relative length of C as

$$\operatorname{dist}(A_0, A_1, C) = \int_C \frac{\|dx\|}{\|x\|} := \int_C \frac{\|x'(t)\|}{\|x(t)\|}\, dt.$$

In particular,

$$\operatorname{dist}(A_0, A_1, -)$$

denotes the relative length of a straight line segment from A_0 to A_1.

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0021-9045/89 $3.00

J. D. Pryce [3] defined the relative distance $\rho(A_0, A_1)$ between two nonzero A_0, A_1 in E as

$$\rho(A_0, A_1) = \inf \operatorname{dist}(A_1, A_2, C)$$

over all paths from A_0 to A_1 and not passing through 0. He proved that ρ is a metric in $E\backslash\{0\}$ and established its usefulness in error theory. The scalar case was investigated earlier by F. W. J. Olver [2]. The essential property in both cases is that ρ is asymptotically equivalent to the usual relative error $\rho_0(A_0, A_1) = \|A_0 - A_1\|/\|A_0\|$; but ρ has nicer properties, in particular the metric inequality $\rho(x, z) \leqq \rho(x, y) + \rho(y, z)$ and the symmetry $\rho(x, y) = \rho(y, x)$. Since $\ln(1+\rho_0) \leqq \rho \leqq -\ln(1-\rho_0)$ [3, Lemma 3.2(a)] it is fairly easy to turn back to ρ_0 after calculations have been done in terms of ρ.

Usually it is not necessary to compute ρ effectively. Nevertheless, for a good understanding one must know ρ at least in the familiar finite-dimensional Banach spaces since matrix and vector manipulation is the obvious first field of application. J. D. Pryce [3] solved this problem in the case of Hilbert norms; in particular he showed that in any $\mathbb{R}_2^n$ $(n \geqq 2)$ there is a geodesic line from any given x to any given y entirely in the plane $x0y$. Essentially, one has only to consider $\mathbb{R}^2$ and Pryce had complete descriptions in this case.

Since the ∞-norm seems as important as the Hilbert-norm in error calculation during matrix and vector manipulation (the other norms may be less interesting) we study $\mathbb{R}_\infty^2$, making use of our previous results in [1] on $\mathbb{R}_\infty^2$. It turns out that the generalization is not obvious at all; except in a relatively simple case (Remark 1.5) there is no clear indication on how to study further $\mathbb{R}_\infty^n$ for $n \geqq 3$. Some ideas, of course, might be useful in a further investigation.

It is quite remarkable that $\rho((1, -1, 0), (-1, 1, 0))$ (in $\mathbb{R}_\infty^3$) is strictly smaller then $\rho((1, -1), (-1, 1))$ (in $\mathbb{R}_\infty^2$) (Example 2.7). This shows that a simple reduction from $\mathbb{R}_\infty^n$ to $\mathbb{R}_\infty^2$ is not possible (unlike the Hilbert case).

In $\mathbb{R}_\infty^3$ we shall consider the regions $(x)^+$, $(x)^-$, $(y)^+$, $(y)^-$, $(z)^+$, $(z)^-$ where $(x)^+ = \{(x, y, z) \in \mathbb{R}_\infty^3 : 0 < x \geqq \max\{|y|, |z|\}\}$ and where the other regions are defined similarly. Pairs like $(x)^+$ and $(x)^-$ are called opposite.

P_x, P_y, and P_z will denote orthogonal projections on the planes $x = 0$, $y = 0$, and $z = 0$, respectively. If C is any path in $\mathbb{R}$ then the meaning of $P_x(C)$, $P_y(C)$, $P_z(C)$ is also clear.

In Section 1 we compute distances between any points A_1 and A_2 both in $(x)^+$ and find minimal paths between them. (There is a significant amount of symmetry in the situation and it would be boring to mention explicitly how this is exploited in every particular problem; usually geometric intuition shows the way. Here $(x)^+$ is typical for the six regions defined above.) The results of this section may be generalized easily to the case $n \geqq 3$.

We then divide $\mathbb{R}^3_\infty \setminus \{0\}$ further into 24 second-order regions, a typical one of which is $(x)^+ (y)^+ := \{(x, y, z): 0 < x \geqq y = |y| \geqq |z|\}$. Pairs like $(x)^+ (y)^+$ and $(x)^- (y)^-$ are called opposite; pairs like $(x)^+ (y)^+$ and $(y)^- (x)^-$ semi-opposite.

In Section 2 we consider points A_0, A_1 in adjacent regions, e.g., $A_0 \in (x)^+$, $A_1 \in (y)^+$. If they are not in semi-opposite second-order regions, then the exact nature of a minimal path from A_0 to A_1 is determined. If one of A_0, A_1 is in a second-order region that includes $(x)^+ \cap (y)^+$, then the actual parameters are also obtained. If this is not the case but A_0 and A_1 are in the same half-space determined by the $x0y$ plane, then computation of the parameters is reduced to minimizing a rational function, with a numerator of degree 3 and a denominator of degree 2, over a half-line. In other cases, which are practically of little interest, more complicated minimizations may be necessary.

In Section 3 points in opposite regions are considered, e.g., $A_0 \in (x)^+$, $A_1 \in (x)^-$. Depending on the second-order region, a minimal path is found after solving at most four minimization problems (independent of each other) in at most six variables.

For practical purposes points in opposite regions or in semi-opposite second-order regions seem rather uninteresting (this would mean a very poor approximation indeed) so that the above results are satisfying.

1. The Shortest Path in a Region of Type $(x)^+$

Let A_0 and A_1 be any points in $(x)^+$. We shall find a minimal path from A_0 to A_1 that lies entirely in $(x)^+$. For obvious symmetry reasons, similar results hold in the five other regions.

Proposition 1.1. *Let $A_0, A_1 \in (x)^+$. Then for every path C from A_0 to A_1 there is a path C_1 from A_0 to A_1 entirely in $(x)^+$ and such that* $\operatorname{dist}(A_0, A_1, C_1) \leqq \operatorname{dist}(A_0, A_1, C)$.

Proof. Let C be determined by $[\alpha, \beta] \to \mathbb{R}^3_\infty$, $t \mapsto C(t)$. Consider $i, j, k, l: \mathbb{R}^3_\infty \to \mathbb{R}^3_\infty$ where $i(x, y, z) = (x, y, z)$ if $y \leqq x$, $i(x, y, z) = (y, x, z)$ if $y > x$, $j(x, y, z) = (x, y, z)$ if $-y \leqq x$, $j(x, y, z) = (-y, -x, z)$ if $-y > x$, $k(x, y, z) = (x, y, z)$ if $z \leqq x$, $k(x, y, z) = (z, y, x)$ if $z > x$, $l(x, y, z) = (x, y, z)$ if $-z \leqq x$, $l(x, y, z) = (-z, y, -x)$ if $-z > x$. Then $l \circ k \circ j \circ i \circ C$ is the requested path C_1.

Proposition 1.2. *Let $A_0 = (x_0, y_0, z_0)$, $A_1 = (x_1, y_1, z_1) \in \mathbb{R}^3_\infty$.*

(α) *If the straight line segment* A_0-A_1 *lies entirely in* $\{(x, y, z)$: $|z| \leqq \max\{|x|, |y|\}\}$, *then* $\operatorname{dist}(A_0, A_1, -) \geqq \operatorname{dist}(P_z A_0, P_z A_1, -)$.

(β) *If, moreover,* $|z_1-z_0| \leqq \max\{|x_1-x_0|, |y_1-y_0|\}$ *then equality holds above.*

Proof. Recall that P_z denotes orthogonal projection on the $x0y$ plane. By their definitions $\operatorname{dist}(A_0, A_1, -)$ and $\operatorname{dist}(P_z A_0, P_z A_1, -)$ are given as integrals of fractions. In corresponding points these fractions have the same denominators. In the case (β) they also have the same numerator; otherwise the numerator in $\operatorname{dist}(A_0, A_1, -)$ is at least not smaller than that in $\operatorname{dist}(P_z A_0, P_z A_1, -)$.

Remark 1.3. The condition in (α) above holds in particular if $A_0, A_1 \in (x)^+$.

PROPOSITION 1.4. *Let* $A_0=(x_0, y_0, z_0)$, $A_1=(x_1, y_1, z_1) \in (x)^+$. *There is a minimal path from* A_0 *to* A_1 *that lies entirely in* $(x)^+$. *If* $|y_1-y_0| \geqq \max\{|x_1-x_0|, |z_1-z_0|\}$ *then it consists of two line segments. It is obtained via a minimal path* $t \mapsto (x(t), y(t))$ *from* $P_z(A_0)$ *to* $P_z(A_1)$ *in the plane* $z=0$ *by adding a third coordinate* $z(t)=z_0+((y(t)-y_0)/(y_1-y_0))(z_1-z_0)$. *It is obtained similarly if* $|z_1-z_0| \geqq \max\{|x_1-x_0|, |y_1-y_0|\}$. *Finally, it reduces to one line segment if* $|x_1-x_0| \geqq \max\{|y_1-y_0|, |z_1-z_0|\}$.

Proof. We consider only the case $|y_1-y_0| \geqq \max\{|x_1-x_0|, |z_1-z_0|\}$, the other situations being similar. Essentially we prove that the path which is described above has a length smaller than or equal to the length of any other path C from A_0 to A_1 that consists of a finite number of straight line segments entirely in $(x)^+$. This is sufficient in view of Proposition 1.1 and an easy approximation argument. From Proposition 1.2(α) and Remark 1.3 we know that

$$\operatorname{dist}(A_0, A_1, C) \geqq \operatorname{dist}_{z=0}(P_z(A_0), P_z(A_1), P_z(C)),$$

where $\operatorname{dist}_{z=0}$ is the distance computed in the plane $z=0$. Also, from [1, Proposition 2.2] we know that a shortest path in the plane $z=0$ from $P_z(A_0)$ to $P_z(A_1)$ exists and lies in $(x)^+ \cap \{(x, y, z)\colon z=0\}$. (Actually, it is a line consisting of two segments, namely

$$(x_0, y_0) \to \left(\frac{x_0+x_1}{2}+\frac{y_1-y_0}{2}, \frac{x_1-x_0}{2}+\frac{y_1+y_0}{2}\right) \to (x_1, y_1)$$

if we assume $y_0 < y_1$.)

Let $C_0\colon t \mapsto (x(t), y(t))$ be such a minimal path. Define C_1 by adding a parameter function $t \mapsto z(t)=z_0+((y(t)-y_0)/(y_1-y_0))(z_1-z_0)$. Then

$|z'(t)| \leqq |y'(t)|$ and so $\operatorname{dist}(A_0, A_1, C) \geqq \operatorname{dist}_{z=0}(P_z(A_0), P_z(A_1), P_z(C)) \geqq \operatorname{dist}_{z=0}(P_z(A_0), P_z(A_1), C_0) \geqq \operatorname{dist}(A_0, A_1, C_1)$.

Remark 1.5. Proposition 1.4 may be generalized easily to the case $n \geqq 3$. If $A_0 = (x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)})$ and $A_1 = (x_1^{(1)}, x_2^{(1)}, ..., x_n^{(1)}) \in \mathbb{R}_\infty^n$ with $0 \leqq x_1^{(j)} \geqq \max_{i \geqq 2} |x_i^{(j)}|$ for $j = 0, 1$, then there exists a minimal path from A_0 to A_1 that lies entirely in $\{(x_i)_{i=1}^n : 0 < x_1 \geqq \max_{i \geqq 2} |x_i|\}$. If k is such that $|x_k^{(0)} - x_k^{(1)}| \geqq \max_{i \neq k} |x_i^{(0)} - x_i^{(1)}|$ then such a minimal path may be found via a projection on the (x_1, x_k)-plane; it consists of no more than two straight line segments. The proof is carried out by generalizing the results in this section in a straightforward manner.

2. The Shortest Path between Two Points in Adjacent Regions

Let $A_0 = (x_0, y_0, z_0) \in (x)^+$ and $A_1 = (x_1, y_1, z_1) \in (y)^+$. We study three types of paths from A_0 to A_1:

(1) Type $(z)^+$: Goes from A_0 to a point $A_2 = (x_2, y_2, x_2) \in (x)^+ \cap (z)^+$, then to a point $A_3 = (x_3, x_3, z_3) \in (z)^+$ $(x_3 > 0)$, to $A_4 = (x_4, y_4, y_4) \in (z)^+ \cap (y)^+$, and finally to A_1. (Here A_3 is introduced only for technical reasons which will become clear in the subsequent proofs.)

(2) Type O: Goes from A_0 to a point $A_3 = (x_3, x_3, z_3) \in (x)^+ \cap (y)^+$ and then to A_1.

(3) Type $(z)^-$: Like $(z)^+$ but with $(z)^+$ replaced by $(z)^-$ throughout.

We first prove that we need only consider these types to find a minimal path from A_0 to A_1. Consider any path $C: [a, b] \to \mathbb{R}^3$ that connects A_0 with A_1. Apply the map j of Proposition 1.1 to show that we may assume

$$C(t) \in \{(x, y, z) \in \mathbb{R}_\infty^3 : x + y \geqq 0\} \subseteq (x)^+ \cup (y)^+ \cup (z)^+ \cup (z)^- \qquad \text{for all } t.$$

Put $t_1 = \sup\{t: C(t) \in (x)^+\}$. If $t_1 = b$, then the path is of Type 0 with $A_3 = A_1$. If $t_1 < b$, then $C(t_1)$ is in one of the regions $(y)^+$, $(z)^+$, or $(z)^-$. If $C(t_1) \in (y)^+$ we apply Proposition 1.1 in $(y)^+$ to get a path of Type O with $A_3 = C(t_1)$.

If $C(t_1) \notin (y)^+$ we may assume (because of symmetry) that $C(t_1) \in (z)^+$. Put $t_2 = \sup\{t: C(t) \in (z)^+\}$. We have $C(t) \in (y)^+ \cup (z)^-$ for all $t > t_2$ and also for $t = b$. So $C(t_2) \in (z)^+ \cap ((y)^+ \cup (z)^-) = (z)^+ \cap (y)^+$. Apply Proposition 1.1 twice, namely on $C(t)$ $(t_1 \leqq t \leqq t_2)$ in $(z)^+$ and $C(t)$ $(t_2 \leqq t \leqq b)$ in $(y)^+$ to get a path connecting A_0 with $A_2 = C(t_1) \in (x)^+ \cap (z)^+$, $A_4 = C(t_2) \in (y)^+ \cap (z)^+$, and A_1. Apply the map j of Proposition 1.1 again to get a path entirely in the half-space $x + y \geqq 0$. This one is of Type

$(z)^+$ where the existence of $A_3=(x_3, x_3, z_3)\in(z)^+$ with $x_3>0$ follows from a continuity argument.

We define $\rho_{z^+}(A_0, A_1)$ as the infimum of the lengths of all Type $(z)^+$-paths from A_0 to A_1; $\rho_O(A_0, A_1)$ and $\rho_{z^-}(A_0, A_1)$ are defined similarly.

There are paths that belong to both Type $(z)^+$ and Type O or both Type $(z)^-$ and Type O. Type $(z)^+$ and Type $(z)^-$ are mutually exclusive.

Results on Type $(z)^+$-paths may be applied to Type $(z)^-$-paths by the symmetry $z\leftrightarrow -z$. We first consider Type O-paths.

PROPOSITION 2.1. *Let $A_0=(x_0, y_0, z_0)\in(x)^+$, $A_1=(x_1, y_1, z_1)\in(y)^+$. Then $\rho_O(A_0, A_1)=2\ln((2x_3+x_0-y_0)(2x_3+y_1-x_1)/4x_3)-\ln(x_0y_1)$ where*

$$x_3=\max\{\tfrac{1}{2}(x_0+y_0), \tfrac{1}{2}(x_1+y_1), \zeta, \tfrac{1}{2}\sqrt{(x_0-y_0)(y_1-x_1)}\}$$

and ζ is the smallest number such that the interval

$$I(\zeta)=[-\zeta, +\zeta]\cap[y_0+z_0-\zeta, z_0-y_0+\zeta]\cap[z_1+x_1-\zeta, z_1-x_1+\zeta]$$

is nonempty.

Moreover, for any $z_3\in I(x_3)$ there is a path connecting A_0 with (x_3, x_3, z_3) and A_1 which has length $\rho_O(A_0, A_1)$.

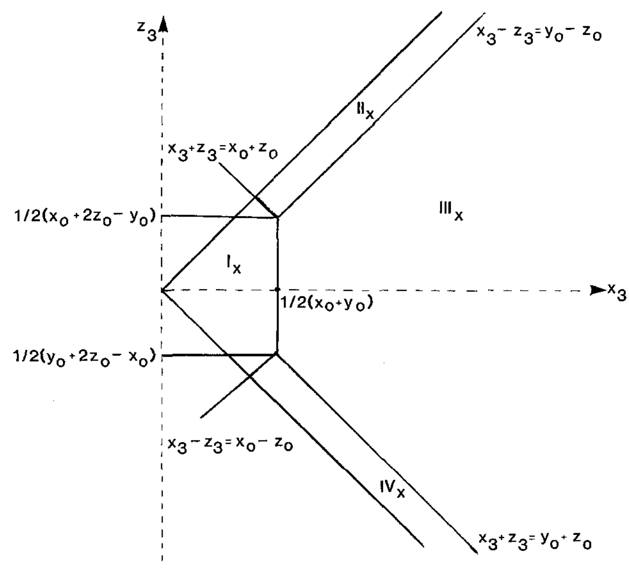

FIGURE 1

Proof. We first compute $\rho(A_0, (x_3, x_3, z_3))$. According to Section 1 we have to compare the quantities $|x_3 - x_0|$, $|x_3 - y_0|$, and $|z_3 - z_0|$. The results of this comparison are presented in Fig. 1. Remark first that $|x_3 - x_0| \leqq |x_3 - y_0|$ iff $x_3 \geqq \frac{1}{2}(x_0 + y_0)$. So the task essentially consists in comparing $|x_3 - y_0|$ with $|z_3 - z_0|$ in the region $x_3 \geqq \frac{1}{2}(x_0 + y_0)$ and in comparing $|x_3 - x_0|$ with $|z_3 - z_0|$ in the region $x_3 \leqq \frac{1}{2}(x_0 + y_0)$. Further details are tedious but straightforward.

It turns out that we must subdivide the $(x)^+$ region of the (x_3, z_3)-plane into the regions I_x, II_x, III_x, and IV_x as in Fig. 1. $|x_3 - x_0|$ dominates in I_x and $|x_3 - y_0|$ dominates in III_x while $|z_3 - z_0|$ dominates in both II_x and IV_x.

Each of these subregions includes its border lines. So intersections are usually nonempty. On the other hand, III_x is the only subregion that is never empty.

Now $\rho(A_0, (x_3, x_3, z_3))$ is given by different formulae in different subregions. As an example, let us consider the region II_x. Here $|z_3 - z_0| \geqq \max\{|x_3 - x_0|, |x_3 - y_0|\}$. Hence by Proposition 1.4 $\rho(A_0, (x_3, x_3, z_3))$ equals $\rho((x_3, z_3), (x_0, z_0))$ in the (x_3, z_3) plane. Since $x_3 + z_3 \geqq x_0 + z_0$ and $x_3 - z_3 \leqq y_0 - z_0 \leqq x_0 - z_0$ a minimal path from (x_3, z_3) to (x_0, z_0) is given by [1, Proposition 2.1]. It consists of the straight line segments $[(x_3, z_3), (\frac{1}{2}(x_3 + x_0 + z_3 - z_0), \frac{1}{2}(x_3 + z_3 - x_0 + z_0))]$ and $[(\frac{1}{2}(x_3 + x_0 + z_3 - z_0), \frac{1}{2}(x_3 + z_3 - x_0 + z_0)), (x_0, z_0)]$ and so has length

$$\ln \frac{(x_3 + z_3 + x_0 - z_0)}{2x_3} + \ln \frac{(x_3 + z_3 + x_0 - z_0)}{2x_0} = \ln \frac{(x_3 + z_3 + x_0 - z_0)^2}{4x_0 x_3}.$$

The other subregions can be studied by similar arguments. We obtain

$$D_{\mathrm{I}}^x = \ln \frac{x_0}{x_3}$$

$$D_{\mathrm{II}}^x = \ln \frac{(x_0 - z_0 + x_3 + z_3)^2}{4x_0 x_3}$$

$$D_{\mathrm{III}}^x = \ln \frac{(x_0 - y_0 + 2x_3)^2}{4x_0 x_3}$$

$$D_{\mathrm{IV}}^x = \ln \frac{(x_0 + z_0 + x_3 - z_3)^2}{4x_0 x_3}$$

whereby $\rho(A_0, (x_3, x_3, z_3)) = D_{\mathrm{I}}^x(x_3, z_3)$ if $(x_3, z_3) \in I_x$ and so on.

Next we compute $\rho(A_1, (x_3, x_3, z_3))$. We have to consider another subdivision of the $(x)^+$-region in the (x_3, z_3)-plane, namely in I_y, II_y, III_y, and

IV_y according to a figure that is obtained by replacing x_0 by y_1, y_0 by x_1, and z_0 by z_1 in Fig. 1. The relative distances $\rho((x_3, x_3, z_3), A_1)$ are given by

$$D_{\mathrm{I}}^y = \ln \frac{y_1}{x_3}$$

$$D_{\mathrm{II}}^y = \ln \frac{(y_1 - z_1 + x_3 + z_3)^2}{4y_1 x_3}$$

$$D_{\mathrm{III}}^y = \ln \frac{(y_1 - x_1 + 2x_3)^2}{4y_1 x_3}$$

$$D_{\mathrm{IV}}^y = \ln \frac{(y_1 + z_1 + x_3 - z_3)}{4y_1 x_3}.$$

We now have to minimize $\rho(A_0, (x_3, x_3, z_3)) + \rho(A_1, (x_3, x_3, z_3))$ in the $(x)^+$-region of (x_3, z_3)-space. We claim that the minimum value of $\rho(A_0, (x_3, x_3, z_3)) + \rho(A_1, (x_3, x_3, z_3))$ is obtained in a point of $\mathrm{III}_x \cap \mathrm{III}_y$. To prove this, we eliminate successively the 15 other intersections of subregions which compete with $\mathrm{III}_x \cap \mathrm{III}_y$. Note that each intersection contains its border lines (but not the point 0) and that the order of elimination is essential. We proceed as follows:

$\mathrm{I}_x \cap \mathrm{I}_y$: Let x_3 increase. Then $D_{\mathrm{I}}^x + D_{\mathrm{I}}^y$ decrease until we reach the border line with another region.

$\mathrm{I}_x \cap \mathrm{II}_y$: Let x_3 increase by fixed $x_3 + z_3$. Remark that we do not reënter $\mathrm{I}_x \cap \mathrm{I}_y$ and that we do not enter $\mathrm{I}_x \cap \mathrm{IV}_y$.

$\mathrm{I}_x \cap \mathrm{IV}_y$: Let x_3 increase by fixed $x_3 - z_3$.

$\mathrm{II}_x \cap \mathrm{I}_y$ and $\mathrm{IV}_x \cap \mathrm{I}_y$: Similar.

$\mathrm{I}_x \cap \mathrm{III}_y$ and $\mathrm{III}_x \cap \mathrm{I}_y$: Let x_3 increase.

We have now eliminated $\mathrm{I}_x \cup \mathrm{I}_y$ (except the border lines) and must not reenter it.

$\mathrm{II}_x \cap \mathrm{II}_y$, $\mathrm{II}_x \cap \mathrm{IV}_y$, $\mathrm{IV}_x \cap \mathrm{II}_y$, $\mathrm{IV}_x \cap \mathrm{IV}_y$: These can similarly be reduced to regions in $\mathrm{III}_x \cap \mathrm{III}_y$.

$\mathrm{II}_x \cap \mathrm{III}_y$: This reduces to $\mathrm{III}_x \cap \mathrm{III}_y$ by keeping $x_3 + z_3$ fixed and increasing x_3 (III_y is stable under this translation!)

$\mathrm{IV}_x \cap \mathrm{III}_y$, $\mathrm{III}_x \cap \mathrm{II}_y$, and $\mathrm{III}_x \cap \mathrm{IV}_y$ are reduced to $\mathrm{III}_x \cap \mathrm{III}_y$ by the same type of arguments.

This proves that we can restrict the minimization problem to $\mathrm{III}_x \cap \mathrm{III}_y$. Put

$$F(x_3) \equiv 2 \ln \frac{(x_0 - y_0 + 2x_3)(y_1 - x_1 + 2x_3)}{4x_3} - \ln(x_0 y_1).$$

Then $\rho(A_0, (x_3, x_3, z_3)) + \rho((x_3, x_3, z_3), A_1) = D^x_{\mathrm{III}} + D^y_{\mathrm{III}} = F(x_3)$ for all $(x_3, x_3, z_3) \in \mathrm{III}_x \cap \mathrm{III}_y$. For $\zeta > 0$ let $I(\zeta)$ be the intersection of $\mathrm{III}_x \cap \mathrm{III}_y$ with the line $x_3 = \zeta$. So

$$\rho_0(A_0, A_1) = \inf\{F(\zeta): \zeta \geqq \max\{\tfrac{1}{2}(x_0 + y_0), \tfrac{1}{2}(x_1 + y_1)\} \quad \text{and} \quad I(\zeta) \neq \phi\}$$

and

$$I(\zeta) = [-\zeta, +\zeta] \cap [y_0 + z_0 - \zeta, z_0 - y_0 + \zeta] \cap [x_1 + z_1 - \zeta, z_1 - x_1 + \zeta]$$

for $\zeta \geqq \max\{\frac{1}{2}(x_0 + y_0), \frac{1}{2}(x_1 + y_1)\}$.

It is easily seen that $F(x_3)$ is decreasing for $0 \leqq x_3 \leqq \frac{1}{2}\sqrt{(x_0 - y_0)(y_1 - x_1)}$, reaches a minimum for $x_3 = \frac{1}{2}\sqrt{(x_0 - y_0)(y_1 - x_1)}$, and is increasing for larger values of x_3. This proves the formula for $\rho_O(A_0, A_1)$ in the statement of Proposition 2.1. Since $\rho(A_0, (x_3, x_3, z_3)) + \rho((x_3, x_3, z_3), A_1))$ does not depend on z_3 if $(x_3, z_3) \in \mathrm{III}_x \cap \mathrm{III}_y$, the additional claim follows as well.

COROLLARY 2.2. *If $y_0 \leqq z_0$, $x_1 \leqq z_1$, then*

$$\rho_O(A_0, A_1) = 2\ln \frac{(2x_3 + x_0 - y_0)(2x_3 + y_1 - x_1)}{4x_3} - \ln(x_0 y_1),$$

where $x_3 = \max\{\frac{1}{2}(x_0 + y_0), \frac{1}{2}(x_1 + y_1), \frac{1}{2}\{\sqrt{(x_0 - y_0)(y_1 - x_1)}\}$.

If $|z_3| \leqq x_3$ and $x_3 + z_3 \geqq \max\{y_0 + z_0, x_1 + z_1\}$ (in particular, if $x_3 - z_3 \leqq \min\{x_0 - z_0, y_1 - z_1\}$), then there is a path of that length through (x_3, x_3, z_3).

Proof. Put $x_3 = \max\{\frac{1}{2}(x_0 + y_0), \frac{1}{2}(x_1 + y_1), \frac{1}{2}\sqrt{(x_0 - y_0)(y_1 - x_1)}\}$. Then $x_3 \in I(x_3)$ since $y_0 + z_0 - x_3 \leqq x_0 + y_0 - x_3 \leqq 2x_3 - x_3 = x_3 \leqq z_0 - y_0 + x_3$ and $z_1 + x_1 - x_3 \leqq x_1 + y_1 - x_3 \leqq 2x_3 - x_3 = x_3 \leqq z_1 - x_1 + x_3$. Hence $I(x_3) \neq \phi$ and the formula for $\rho_O(A_0, A_1)$ is obtained. Also, $z_3 \in I(x_3)$ iff $|z_3| \leqq x_3$ and $z_3 \geqq \max\{y_0 + z_0 - x_3, z_1 + x_1 - x_3\}$, i.e., $x_3 + z_3 \geqq \max\{y_0 + z_0, x_1 + z_1\}$.

Remark that $x_3 - z_3 \leqq x_0 - z_0$ implies $x_3 + z_3 = 2x_3 - (x_3 - z_3) \geqq 2x_3 - x_0 + z_0 \geqq x_0 + y_0 - x_0 + z_0 \geqq y_0 + z_0$.

COROLLARY 2.3. (a) *If $y_0 \leqq z_0$, $x_1 \leqq z_1$, then $\rho_{z^+}(A_0, A_1) \leqq \rho_O(A_0, A_1)$.*

(b) *If $y_0 \leqq -z_0$, $x_1 \leqq -z_1$, then*

$$\rho_{z^-}(A_0, A_1) \leqq \rho_O(A_0, A_1).$$

Proof. For (a) there is a minimal Type O-path from A_0 to A_1 that goes through a point in $(x)^+ \cap (y)^+ \cap (z)^+$ by the preceding corollary. Such a path is, evidently, also a Type $(z)^+$-path. Furthermore, (b) follows from (a) by an argument based on the symmetry $z \leftrightarrow -z$.

We now turn to the exact calculation of Type $(z)^+$-paths of minimal length. The following auxiliary result enables us to find (x_2, y_2) by given (x_3, z_3).

PROPOSITION 2.4. *Let $A_0 = (x_0, y_0, z_0) \in (x)^+$, $y_0 \leqq z_0$, and $A_3 = (x_3, x_3, z_3) \in (z)^+$, $x_3 \geqq 0$. Among the paths $A_0 \to A_2 \to A_3$ where $A_2 = (x_2, y_2, x_2)$ is an arbitrary point in $(x)^+ \cap (z)^+$ there is one with minimal length which satisfies $x_2 - y_2 = z_0 - y_0$. Its length equals*

$$2 \ln \frac{(2x_2 + x_0 - z_0)(2x_2 + z_3 - x_3)}{4x_2} - \ln(x_0 z_3),$$

where

$$x_2 = \max(\tfrac{1}{2}(x_0 + z_0), \tfrac{1}{2}(x_3 + z_3), \tfrac{1}{2}(z_0 - y_0 + 2x_3), \tfrac{1}{2}\sqrt{(x_0 - z_0)(z_3 - x_3)}).$$

Proof. This follows from Proposition 2.1; more precisely it follows from the special case $z_0 \leqq y_0$, $z_1 = x_1 \geqq 0$. In this case $I(\zeta) = [-\zeta, +\zeta] \cap [y_0 + z_0 - \zeta, z_0 - y_0 + \zeta] \cap [2x_1 - \zeta, \zeta] = [y_0 + z_0 - \zeta, z_0 - y_0 + \zeta] \cap [2x_1 - \zeta, \zeta]$. Hence $I(\zeta) \neq \phi$ if and only if $z_0 - y_0 + \zeta \geqq 2x_1 - \zeta$; i.e., $\zeta \geqq 2x_1 + (y_0 - z_0)$ which gives the desired expression in x_2 after the obvious symmetry adaptations ($y \leftrightarrow z$, choice of indices). Since we had $z_3 = z_0 - y_0 + x_3 \in I(x_3)$ the other claim in the proposition ($x_2 - y_2 = z_0 - y_0$) also follows.

PROPOSITION 2.5. *Let $A = (x_0, y_0, z_0) \in (x)^+$, $y_0 \leqq z_0$, and $A_1(x_1, y_1, z_1) \in (y)^+$ with $x_1 \leqq z_1$. Among all Type $(z)^+$-paths from A_0 to A_1 there is one which has minimal length and satisfies*

$$x_2 - y_2 = z_0 - y_0 \tag{1}$$

$$x_2 = \max\{\tfrac{1}{2}(x_0 + z_0), \tfrac{1}{2}(x_3 + z_3), \tfrac{1}{2}(z_0 - y_0 + 2x_3), \tfrac{1}{2}\sqrt{(x_0 - z_0)(z_3 - x_3)}\} \tag{2}$$

$$y_4 - x_4 = z_1 - x_1 \tag{3}$$

$$y_4 = \max\{\tfrac{1}{2}(y_1 + z_1), \tfrac{1}{2}(x_3 + z_3), \tfrac{1}{2}(z_1 - x_1 + 2x_3), \tfrac{1}{2}\sqrt{(y_1 - z_1)(z_3 - x_3)}\}. \tag{4}$$

We have

$$\rho_{z^+}(A_0, A_1) = \inf 2 \ln \frac{(x_0 - y_0 + 2x_3)(y_1 - x_1 + 2x_3)(z_0 - y_0 + z_3 + x_3)(z_1 - x_1 + z_3 + x_3)}{4(z_0 - y_0 + 2x_3)(z_1 - x_1 + 2x_3)z_3} - \ln(x_0, y_1) \tag{5}$$

whereby the inf *is taken over* $0 \leqq x_3 \leqq z_3$ *and*

$$z_0 - y_0 + 2x_3 \geqq \max\{x_0 + z_0, z_3 + x_3, \sqrt{(x_0 - z_0)(z_3 - x_3)}\} \tag{6}$$

$$z_1 - x_1 + 2x_3 \geqq \max\{y_1 + z_1, z_3 + x_3, \sqrt{(y_1 - z_1)(z_3 - x_3)}\}. \tag{7}$$

Proof. Applying Proposition 2.4 twice (once directly and once with interchange $x \leftrightarrow y$) we see that x_2, y_2, x_4, y_4 must satisfy (1), (2), (3), (4). Furthermore, we conclude that

$$\rho_{z^+}(A_0, A_1) = \inf\{2 \ln G(z_3, x_3) \colon 0 \leqq x_3 \leqq z_3 > 0\} - \ln(x_0 y_1), \tag{8}$$

where

$$G(z_3, x_3) = \frac{(2x_2 + x_0 - z_0)(2x_2 + z_3 - x_3)(2y_4 + y_1 - z_1)(2y_4 + z_3 - x_3)}{16 x_2 y_4 z_3}$$

and x_2, y_4 are functions of x_3, z_3 determined by (1)–(4). In the region $0 \leqq x_3 \leqq z_3 > 0$ of the (z_3, x_3)-plane we shall now consider different regions:

$$D_1^2 := \{(z_3, x_3) \mid x_2 = \tfrac{1}{2}(x_0 + z_0)\}$$
$$D_2^2 := \{(z_3, x_3) \mid x_2 = \tfrac{1}{2}(x_3 + z_3)\}$$
$$D_3^2 := \{(z_3, x_3) \mid x_2 = \tfrac{1}{2}(z_0 - y_0 + 2x_3)\}$$
$$D_4^2 := \{(z_3, x_3) \mid x_2 = \tfrac{1}{2}\sqrt{(x_0 - z_0)(z_3 - x_3)}\}.$$

All these regions are topologically closed except, possibly, for a boundary point $(0, 0)$; some of them may be empty. In general the intersections may be empty; if they are nonempty they consist of segments of lines or of quadratic curves. In any case $D_1^2 \cup D_2^2 \cup D_3^2 \cup D_4^2 = \{(z_3, x_3) : 0 \leqq x_3 \leqq z_3 > 0\}$. Analogous remarks apply to another system of subsets of $\{(z_3, x_3)$: $0 \leqq x_3 \leqq z_3 > 0\}$

$$D_1^4 := \{(z_3, x_3) \mid y_4 = \tfrac{1}{2}(y_1 + z_1)\}$$
$$D_2^4 := \{(z_3, x_3) \mid y_4 = \tfrac{1}{2}(x_3 + z_3)\}$$
$$D_3^4 := \{(z_3, x_3) \mid y_4 = \tfrac{1}{2}(z_1 - x_1 + 2x_3)\}$$
$$D_4^4 := \{(z_3, x_3) \mid y_4 = \tfrac{1}{2}\sqrt{(y_1 - z_1)(z_3 - x_3)}\}.$$

To obtain an explicit formula for $\rho_{z^+}(A_0, A_1)$ we consider $G(z_3, x_3)$ separately in each of the 16 regions $D_{i,j}$ $(1 \leqq i, j \leqq 4)$ where $D_{i,j} = D_i^2 \cap D_j^4$. In each region $D_{i,j}$ we replace x_2, y_4 by the corresponding formulae in the definitions of D_i and D_j. We then minimize $G(z_3, x_3)$ in $D_{i,j}$. It turns out that in each $D_{i,j}$ the infimum is reached in a border point. Actually, for

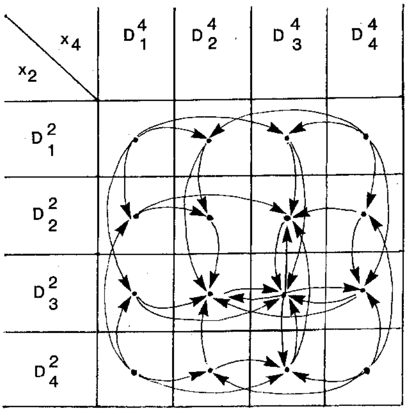

FIGURE 2

every region $D_{i,j}$ we find a number of other regions, say $\{(D_{i(k),j(k)}: 1 \leqq k \leqq n\}$, where $1 \leqq n \leqq 4$ such that for every $(z_3, x_3) \in D_{i,j}$ there exists a k $(1 \leqq k \leqq n)$ and a point $(z'_3, x'_3) \in D_{i(k),j(k)}$ with $G(z_3, x_3) \geqq G(z'_3, x'_3)$. Of course we require $(i(k), j(k)) \neq (i,j)$ for all k $(1 \leqq k \leqq n)$.

The results of this investigation are given in Fig. 2 by means of arrows which lead from $D_{i,j}$ to $D_{i(k),j(k)}$ for all possible i, j, k. We shall not give too many details on the calculations here; it is to be remarked, however, that for symmetry reasons we need not investigate 16 cases but only $4+(16–4):2=10$. As an example consider $D_{3,3}$ (even a glance at Fig. 2 shows that this region is particularly interesting; also it is never empty because a point (z_3, x_3) with $z_3 = x_3$ large enough will belong to it).
Then

$$G(z_3, x_3) = \frac{(x_0 - y_0 + 2x_3)(z_0 - y_0 + x_3 + z_3)(y_1 - x_1 + 2x_3)(z_1 - x_1 + z_3 + x_3)}{4(z_0 - y_0 + 2x_3)(z_1 - x_1 + 2x_3)\, z_3}.$$

For fixed x_3 this function behaves like

$$\frac{(z_0 - y_0 + x_3 + z_3)(z_1 - x_1 + z_3 + x_3)}{z_3}. \tag{9}$$

Now remark that in $D_{3,3}$ we have

$$z_3 \leqq z_0 - y_0 + x_3 \tag{10}$$

$$z_3 \leqq z_1 - x_1 + x_3 \tag{11}$$

so that

$$z_3 \leqq \sqrt{(z_0 - y_0 + x_3)(z_1 - x_1 + x_3)}. \tag{12}$$

This implies that (9) is a decreasing function of z_3. Hence z_3 may be replaced by greater values until we meet another region; a look at the definitions of the D_j^i shows that this can only be D_2^2, D_4^2, D_2^4, or D_4^4. Therefore Fig. 2 displays four arrows departing from $D_{3,3}$, namely to $D_{2,3}, D_{4,3}, D_{32}$, and $D_{3,4}$. Remark that in practice it is quite possible to decide which of them has to be considered as soon as we know $x_0, y_0, z_0, x_1, y_1, z_1$.

An inspection of Fig. 2 proves now that for every (z_3, x_3) $(0 \leqq x_3 \leqq z_3 > 0)$ there is a couple $(z_3', x_3' \in D_{3,3}$ which relates to a path with shorter or equal length. Since (6) and (7) just express that $(z_3, x_3) \in D_{3,3}$ and since the expression in (5) is that obtained by replacing, in (8), x_2 and y_4 by their values in $D_{3,3}$, the proof is complete.

Remark 2.6. (a) As suggested in the proof of Proposition 2.5 the minimizing problem posed in (5), (6), (7) is usually easier than it looks since by fixed x_3 we find z_3 as the maximal number which satisfies (6), (7).

(b) If $z_0 > 0$, then we have $\sqrt{(x_0 - z_0)(z_3 - x_3)} \leqq \frac{1}{2}(x_0 - z_0) + \frac{1}{2}(z_3 - x_3) \leqq \sup\{x_0 + z_0, z_3 + x_3\}$ so that (6) simplifies. Actually, if both $z_0, z_1 > 0$, then (6) and (7) are equivalent to

$$x_3 \geqq \tfrac{1}{2}\max\{x_0 + y_0, y_1 + x_1\}$$

$$z_3 \leqq \min\{z_0 - y_0, z_1 - x_1\} + x_3$$

so that we may assume

$$z_3 - x_3 = \min\{z_0 - y_0, z_1 - x_1\}.$$

If in this case $z_0 - y_0 \leqq z_1 - x_1$ then (5) reduces to

$$\begin{aligned}\rho_{z^+}(A_0, A_1) &\\ &= \inf 2 \ln \frac{(x_0 - y_0 + 2x_3)(z_0 - y_0 + z_1 - x_1 + 2x_3)(y_1 - x_1 + 2x_3)}{2(z_0 - y_0 + 2x_3)(z_1 - x_1 + 2x_3)}\\ &\quad - \ln(x_0 y_1),\end{aligned}$$

where the inf is taken over $x_3 \geqq \frac{1}{2}\max\{x_0 + y_0, y_1 + x_1\}$. Since the resulting formula is symmetric, it also holds if $z_0 - y_0 \geqq z_1 - x_1$.

(c) In the worst case we have $z_0 - y_0 + 2x_3 = \sqrt{(x_0 - z_0)(z_3 - x_3)}$ or $z_1 - x_1 + 2x_3 = \sqrt{(y_1 - z_1)(z_3 - x_3)}$. Then in (5) we have to minimize a rational function in x_3 whose numerator has degree $\leqq 6$ and whose denominator has degree $\leqq 4$.

EXAMPLE 2.7. Take $A_0=(1,-1,0)$, $A_1=(-1,1,0)$. From [1] we know that $\rho((1,-1),(-1,1))=4\ln 2$ in $\mathbb{R}^2_\infty$. Applying Proposition 2.5 and taking Remark 2.6(b) into account we find $z_3-x_3=1$. So

$$\rho_{z^+}(A_0,A_1)=\inf_{x_3\geqq 0} 2\ln\frac{(2+2x_3)^4}{4(1+2x_3)^2\,(1+x_3)}$$

$$=\inf_{x_3\geqq 0} 2\ln\frac{4(1+x_3)^3}{(1+2x_3)^2}.$$

A routine calculation shows that the inf is attained for $x_3=\frac{1}{2}$ so that $z_3=\frac{3}{2}$ and

$$\rho_{z^+}(A_0,A_1)=6\ln\tfrac{3}{2}.$$

Using (1)–(4) we also obtain

$$x_2=\max\{\tfrac{1}{2},1,1,\sqrt{1.1}\}=1$$

$$y_2=x_2-(z_0-y_0)=1-1=0$$

$$y_4=1$$

$$x_4=0.$$

The shortest path so goes successively through $A_0=(1,-1,0)$, $(1,0,1)$, $(\frac{1}{2},\frac{1}{2},\frac{3}{2})$, $(0,1,1)$, and $A_1=(-1,1,0)$. The most remarkable fact is that $\rho(A_0,A_1)$ $(=6\ln\frac{3}{2})$ is strictly smaller than the distance from A_0 to A_1 as calculated in the (x,y)-plane $\mathbb{R}^2_\infty$ $(=4\ln 2)$.

PROPOSITION 2.8. *If either $y_0-z_0\geqq 0$ or $x_1-z_1\geqq 0$, then*

$$\rho_{z^+}(A_0,A_1)\geqq\rho_O(A_0,A_1).$$

Proof. By symmetry, we may assume that $y_0-z_0\geqq 0$. Essentially we show that for every Type $(z)^+$-path from A_0 to A_1 there is a Type O-path C_1 from A_0 to A_1 with equal or smaller length. Let C be a Type $(z)^+$-path from A_0 to A_1. Since $y_0-z_0\geqq 0$ and $y_2-x_2\leqq 0$ the path C meets a point of type $A_5=(x_5,y_5,y_5)\in(x)^+$. We apply Proposition 2.4 to the restriction of C to a path from A_5 to A_3. It follows that we may assume $x_2-y_2=0$. This means that $A_2\in(x)^+\cap(y)^+$. Since A_2, A_1 both lie in $(y)^+$ it follows from Proposition 1.1 that we are reduced to a Type O-path.

COROLLARY 2.9. *If either $y_0+z_0\geqq 0$ or $x_1+z_1\geqq 0$, then*

$$\rho_{z^-}(A_0,A_1)\geqq\rho_O(A_0,A_1)$$

(by duality $z\leftrightarrow -z$).

$(x)^+ \backslash (y)^+$	$(z)^+$	$(x)^-$	$(x)^+$	$(z)^-$
$(z)^+$	$(z)^+$	$(z)^+$	O	O
$(y)^-$	$(z)^+$	$(z)^+$ or $(z)^-$	O	$(z)^-$
$(y)^+$	O	O	O	O
$(z)^-$	O	$(z)^-$	O	$(z)^-$

FIGURE 3

Conclusion. The results of this section are collected in Fig. 3 where, for example, the O in the third row and second column indicates that there is a Type O-path from $A_0 \in (x)^+ (y)^+$ to $A_1 \in (y)^+ (x)^-$ with length $\rho(A_0, A_1)$. If the type is O, then this is true by virtue of Proposition 2.8 and 2.9; if it is $(z)^+$ then by 2.9 and 2.3(a); if it is $(z)^-$, then by 2.8 and 2.3(b).

A difficulty arises if $A_0 \in (x)^+ (y)^-$ and $A_1 \in (y)^+ (x)^-$ (semi-opposite second-order regions). If in this case $z_0 \geqq 0$, $z_1 \geqq 0$, then by a symmetry argument we are in Type $(z)^+$; if $z_0 \leqq 0$, $z_1 \leqq 0$, then similarly we are in Type $(z)^-$.

Type O may be excluded always by Corollary 2.3. The case where $z_0 z_1 < 0$ (which is of little interest anyway since the approximation is very poor) must be solved by actual calculations using Proposition 2.5.

3. THE SHORTEST PATH BETWEEN TWO POINTS IN OPPOSITE REGIONS

Let A_0, A_1 be points in opposite regions. By symmetry we may assume $A_0 \in (x)^+ (z)^+$, $A_1 \in (x)^-$. We distinguish three cases, namely, (i) $A_1 \in (x)^- (z)^+$; (ii) $A_1 \in (x)^- (y)^+$, and (iii) $A_1 \in (x)^- (z)^-$. The case $A_1 \in (x)^- (y)^-$ indeed reduces by symmetry to (ii). In each of these cases we formulate a proposition on the nature of a shortest path from A_0 to A_1.

PROPOSITION 3.1. *If $A_0 \in (x)^+ (z)^+$ and $A_1 \in (x)^- (z)^+$, then there is a minimal path from A_0 to A_1 that successively meets $(x)^+ \cap (z)^+$ and $(x)^- \cap (z)^+$.*

Proof. We may assume that $z \geqq 0$ along the whole path. Put $t_f := \sup\{t: C(t) \in (x)^+\}$. If $C(t_f) \in (x)^+ \cap (z)^+$, then the result follows from an application of the results of Section 2 to the two-region $(z)^+ \cup (x)^-$. If $C(t_f) \in (x)^+ \cap (y)^+$, then we look at $(y)^+ \cup (x)^-$, and if $C(t_f) \in (x)^+ \cap (y)^-$, then we look at $(x)^- \cup (y)^-$.

PROPOSITION 3.2. *If $A_0 \in (x)^+\ (z)^+$ and $A_1 \in (x)^-\ (y)^+$ then there is a shortest path from A_0 to A_1 that goes succesively through $(x)^+ \cap (z)^+$, $(y)^+ \cap (z)^+$, and $(y)^+ \cap (x)^-$.*

Proof. We may assume that $y + z \geqq 0$ along the whole path. Put $t_f := \sup\{t: C(t) \in (x)^+\}$; hence $C(t_f) \in (x)^+ \cap (z)^+$ or $C(t_f) \in (x)^+ \cap (y)^+$. In the first case we apply the results of Section 2 to the two-region $(z)^+ \cup (x)^-$. In the second case first consider the two-region $(x)^- \cup (y)^+$. It follows that the path meets $(y)^+ \cap (x)^-$. Then look at $(x)^+ \cup (y)^+$ to obtain the conclusion.

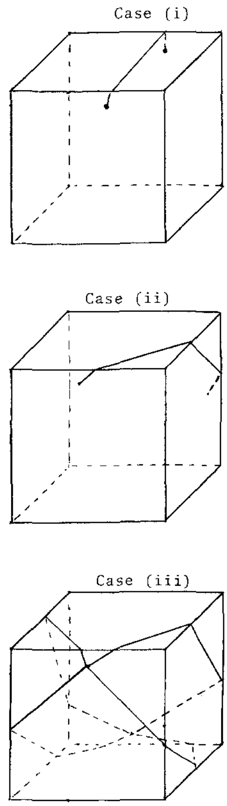

FIGURE 4

PROPOSITION 3.3. *If $A_0 \in (x)^+ \ (z)^+$ and $A_1 \in (x)^- \ (z)^-$, then there is a shortest path $A_0 \to A_1$ of one of the following four types*:

$$\text{(I)}: \quad A_0 \to (x)^+ \cap (z)^+ \to (y)^- \cap (z)^+ \to (y)^- \cap (x)^- \to A_1,$$
$$\text{(II)}: \quad A_0 \to (x)^+ \cap (z)^+ \to (y)^+ \cap (z)^+ \to (y)^+ \cap (x)^- \to A_1,$$
$$\text{(III)}: \quad A_0 \to (x)^+ \cap (y)^+ \to (y)^+ \cap (z)^- \to (x)^- \cap (z)^- \to A_1,$$
$$\text{(IV)}: \quad A_0 \to (x)^+ \cap (y)^- \to (y)^- \cap (z)^- \to (x)^- \cap (z)^- \to A_1.$$

Proof. Again, put $t_f := \sup\{t\colon C(t) \in (x)^+\}$. If $C(t_f) \in (x)^+ \cap (z)^+$, then an application of the results of Section 2 to $(z)^+ \cup (x)^-$ shows that we are either in (I) or in (II). If $C(t_f) \in (x)^+ \cap (y)^+$, then a look at the two-region $(y)^+ \cup (x)^-$ shows that we are in case (III); similarly for $C(t_f) \in (x)^+ \cap (y)^-$ and case (IV).

If $C(t_f) \in (x)^+ \cap (z)^-$ then by considering $(z)^- \cup (x)^-$ we see that the path meets $(x)^- \cap (z)^-$. If we then consider $(x)^+ \cup (z)^-$ it follows that we are in (III) or (IV).

Conclusion. In case (i) we are reduced to a minimization problem in four variables; in case (ii) to one in six variables, in case (iii) to four problems in six variables. (See Figure 4.) The exact nature of the functions that are to be minimized is, of course, given in Section 1.

ACKNOWLEDGMENT

The author thanks the referee who pointed out an error in the original manuscript and made some further useful suggestions.

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